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In any triangle ABC, prove that

$$\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\right)\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right) \geq \frac{9\sqrt{3}}{2}.$$

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Let r and s be, respectively, inradius and semiperimeter of $\triangle ABC$.

Since $\sum\cot\frac{A}{2} = \sum\frac{s-a}{r} = \frac{s}{r} = \dfrac{4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = \prod\cot\frac{A}{2}$

then $\sum\sin\frac{A}{2}\cdot\sum\cot\frac{A}{2} \geq \frac{9\sqrt{3}}{2} \Leftrightarrow \sum\sin\frac{A}{2}\cdot\prod\cot\frac{A}{2} \geq \frac{9\sqrt{3}}{2} \Leftrightarrow$

(**1**) $\quad\sum\sin\frac{A}{2} \geq \frac{9\sqrt{3}}{2}\prod\tan\frac{A}{2}$

Let $\alpha := \frac{\pi-A}{2}, \beta := \frac{\pi-B}{2}, \gamma := \frac{\pi-C}{2}$. Then $\alpha,\beta,\gamma > 0, \alpha+\beta+\gamma = \pi$ and inequality (**1**) becomes

(**2**) $\quad\sum\cos\alpha \geq \frac{9\sqrt{3}}{2}\prod\cot\alpha.$

Let a,b,c be sidelengths of some triangle T with correspondent angles α,β,γ and let R,r,s be, respectively, circumradius, inradius and semiperimeter of this triangle (R,r,s are local notations here for new triangle T).

Since $\sum\cos\alpha = 1+\frac{r}{R}$, $\prod\cot\alpha = \dfrac{s^2-(2R+r)^2}{2sr}$ then (**2**)$\Leftrightarrow$

(**3**) $\quad 1+\frac{r}{R} \geq \dfrac{9\sqrt{3}\left(s^2-(2R+r)^2\right)}{4sr} \Leftrightarrow \dfrac{R+r}{R} \geq \dfrac{9\sqrt{3}\left(s^2-(2R+r)^2\right)}{4sr}.$

Noting that $\dfrac{s^2-(2R+r)^2}{s} = s-\dfrac{(2R+r)^2}{s}$ increase by s and $s^2 \leq 4R^2+4Rr+3r^2$ (Gerretsen's Inequality) we obtain that

$$\frac{s^2-(2R+r)^2}{4sr} \leq \frac{4R^2+4Rr+3r^2-(2R+r)^2}{4r\sqrt{4R^2+4Rr+3r^2}} =$$

$$\frac{2r^2}{4r\sqrt{4R^2+4Rr+3r^2}} = \frac{r}{2\sqrt{4R^2+4Rr+3r^2}}.$$

Thus, remains to prove inequality $\dfrac{R+r}{R} \geq \dfrac{9\sqrt{3}\cdot r}{2\sqrt{4R^2+4Rr+3r^2}} \Leftrightarrow$

$2(R+r)\sqrt{4R^2+4Rr+3r^2} \geq 9\sqrt{3}\cdot Rr.$

Since $R \geq 2r$ (Euler's Inequality) we have

$4(R+r)^2(4R^2+4Rr+3r^2)-243R^2r^2 = (R-2r)(16R^3+80R^2r-23Rr^2-6r^3) \geq 0$

$(16R^3+80R^2r-23Rr^2-6r^3 > 14R^2r-24Rr^2-8r^3 = 2r(R-2r)(7R+2r)).$